

On Universal Tilers

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Abstract

A famous problem in discrete geometry is to find all monohedral plane tilers, which is still open to the best of our knowledge. This paper concerns with one of its variants that to determine all convex polyhedra whose every cross-section tiles the plane. We call such polyhedra universal tilers. We obtain that a convex polyhedron is a universal tiler only if it is a tetrahedron or a pentahedron.

Keywords: cross-section, Euler's formula, pentahedron, universal tiler

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1 Introduction

A monohedral tiler is a polygon that can cover the plane by congruent repetitions without gaps or overlaps. The problem of determining all monohedral tilers, also called the problem of tessellation, was brought anew into mathematical prominence by Hilbert when he posed it as one of his "Mathematische Probleme", see Kershner [4]. It is well-known that all triangles and all quadrangles are tilers. Reinhardt [8] determined all hexagonal tilers, and obtained some special kinds of pentagonal tilers. Later it is shown that any polygon with at least 7 edges is not a tiler by using Euler's formula, see Dress and Huson [3]. The problem of plane tiling, however, is now still open to the best of our knowledge. In fact, there are 14 classes of pentagonal tilers were found, see Hirschhorn and Hunt [7], Sugimoto and Ogawa [9], and Wells [10]. For a whole theory of tessellation patterns, see Grünbaum and Shephard's book [6] as a survey up to 1987.

Considering a variant of the problem of plane tiling, Akiyama [1] found all convex polyhedra whose every development tiles the plane. He call them tile-makers. The main idea in his proof is to investigate the polyhedra whose facets tile the plane by stamping. Notice that facets are special cross-sections. This motivates us to consider a more general class of polyhedron tilers.

Let \mathcal{P} be a convex polyhedron, and π a plane. Denote by $C(\pi)$ the intersection of π and \mathcal{P} . We say that π intersects \mathcal{P} trivially if $C(\pi)$ is empty, or a point, or a line segment. Otherwise we say π intersects \mathcal{P} non-trivially. In this case, $C(\pi)$ is a polygon with at least 3 edges. We call $C(\pi)$ a cross-section if π crosses \mathcal{P} nontrivially. We say that \mathcal{P} is a universal tiler if every cross-section of \mathcal{P} tiles the plane. In this paper, we study the shape of universal tilers. It is a variant of the problem of plane tiling.

It is easy to see that every tetrahedron is a universal tiler since every cross-section of a tetrahedron is either a triangle or a quadrangle. The main goal of this paper is to present that any universal tiler has at most 5 facets.

This paper is organized as follows. In Section 2, we derive a necessary condition that a hexagonal cross-section (if exists) of a universal tiler satisfies. It will be used for excluding the membership of many polyhedra from the class of universal tilers. In Section 3, we prove that any facet of a universal tiler is either a triangle or a quadrangle. In Section 4, by using Euler's formula we obtain that any universal tiler has at most 5 facets.

2 Hexagonal cross-sections of universal tilers

Note that no polygonal tiler has more than 6 edges. It follows that any cross-section of a universal tiler has at most 6 edges. In particular, any facet of a universal tiler has no more than 6 edges. In this section, we shall obtain a necessary condition for hexagonal cross-sections of a universal tiler.

Let \mathcal{P} be a polyhedron, and let π be a plane which crosses \mathcal{P} nontrivially. Let l be a line belonging to π and let $\varepsilon > 0$. Denote by π_+ (resp. π_-) the plane obtained by rotating π around l by the angle ε (resp. $-\varepsilon$). Set $\varepsilon \rightarrow 0$. It is clear that either π_+ or π_- crosses \mathcal{P} nontrivially. Of course it is possible that both π_+ and π_- crosses \mathcal{P} non-trivially. Write

$$p(\pi; l; \varepsilon) = \begin{cases} \pi_+, & \text{if } \pi_+ \text{ crosses } \mathcal{P} \text{ nontrivially;} \\ \pi_-, & \text{otherwise.} \end{cases}$$

Then $p(\pi; l; \varepsilon)$ is a plane crossing \mathcal{P} nontrivially. Intuitively, for small ε , the plane $p(\pi; l; \varepsilon)$ is obtained by rotating the plane π a little along l . For notational simplification, we rewrite

$$C(\pi; l; \varepsilon) = C(p(\pi; l; \varepsilon)).$$

By the continuity of a polyhedron, we see that the cross-section $C(\pi; l; \varepsilon')$ is nontrivial for any $0 < \varepsilon' < \varepsilon$. Let C be a cross-section of \mathcal{P} . We say that C is proper if none of its vertices is a vertex of \mathcal{P} , that is, any vertex of a proper cross-section lies in the interior of an edge of \mathcal{P} .

Lemma 2.1 *If \mathcal{P} has a cross-section with n vertices, then \mathcal{P} has a proper cross-section with at least n vertices.*

Proof. Set up an xyz -coordinate system. For any real number a , denote by π^a the plane determined by the equation $z = a$. Suppose that the cross-section $C(\pi^0)$ has n vertices. Without loss of generality, we can suppose that the half-space $z > 0$ has non-empty intersection with \mathcal{P} . By the continuity of \mathcal{P} , there exists $\delta > 0$ such that for any $0 < \varepsilon < \delta$, the cross-section $C(\pi^\varepsilon)$ has at least n vertices. Consider the z -coordinates of all vertices of \mathcal{P} . Let η be the minimum positive z -coordinate among. Then the cross-section $C(\pi^{\eta/2})$ is a proper cross-section with at least n vertices. This completes the proof. \blacksquare

As will be seen, with aid of the above lemma we may take improper cross-sections out of our consideration. Let $C(\pi) = V_1 V_2 \cdots V_n$ be a proper cross-section of \mathcal{P} . It is clear that for any $1 \leq i \leq n$, there is a unique edge of \mathcal{P} which contains V_i , denoted e_i .

Lemma 2.2 *Let $C(\pi) = V_1 V_2 \cdots V_n$ be a proper cross-section with $V_i \in e_i$. Then there exists $\delta > 0$ such that for any $0 < \varepsilon < \delta$,*

(i) *$C(\pi; V_1 V_3; \varepsilon)$ is a proper cross-section with exactly n vertices which belong to the edges e_1, e_2, \dots, e_n respectively;*

(ii) *if $C(\pi; V_1 V_3; \varepsilon) = U_1^\varepsilon U_2^\varepsilon \cdots U_n^\varepsilon$, where $U_i^\varepsilon \in e_i$, $U_1^\varepsilon = V_1$, and $U_3^\varepsilon = V_3$, then*

$$\angle V_1 U_2^\varepsilon V_3 \neq \angle V_1 V_2 V_3.$$

Proof. Since $C(\pi)$ is proper, by continuity, there exists $\delta_1 > 0$ such that Condition (i) holds for any $0 < \varepsilon < \delta_1$. Suppose that

$$C(\pi; V_1 V_3; \varepsilon) = V_1 U_2^\varepsilon V_3 U_4^\varepsilon U_5^\varepsilon \cdots U_n^\varepsilon,$$

where $U_i^\varepsilon \in e_i$. Let T be the trace of the point U_2^ε as ε varies such that

$$\angle V_1 U_2^\varepsilon V_3 = \angle V_1 V_2 V_3. \quad (2.1)$$

Then T is a sphere if $\angle V_1 V_2 V_3 = \pi/2$, while T is an ellipsoid otherwise. On the other hand, the point U_2^ε moves along e_2 by Condition (i). So U_2^ε belongs to the intersection of a sphere (or ellipsoid) and a line. Such an intersection contains at most two points, say ε_1 and ε_2 . Taking $\delta < \min\{\delta_1, \varepsilon_1, \varepsilon_2\}$, we complete the proof. ■

We need Reinhardt's theorem [8] of the classification of hexagonal tilers. Traditionally, we use the concatenation of two points, say, AB , to denote both the line segment connecting A and B , and its length.

Theorem 2.3 (Reinhardt) *Let $V_1 V_2 \cdots V_6$ be a hexagonal tiler. Then one of the following three properties holds:*

(i) $V_1 + V_2 + V_3 = 2\pi$ and $V_3 V_4 = V_6 V_1$;

(ii) $V_1 + V_2 + V_4 = 2\pi$, $V_2 V_3 = V_4 V_5$ and $V_3 V_4 = V_6 V_1$;

(iii) $V_1 = V_3 = V_5 = 2\pi/3$, $V_2 V_3 = V_3 V_4$, $V_4 V_5 = V_5 V_6$ and $V_6 V_1 = V_1 V_2$.

Figure 1 illustrates the 3 classes of hexagonal tilers. See also Bollobás [2] and Gardner [5] for its proof.

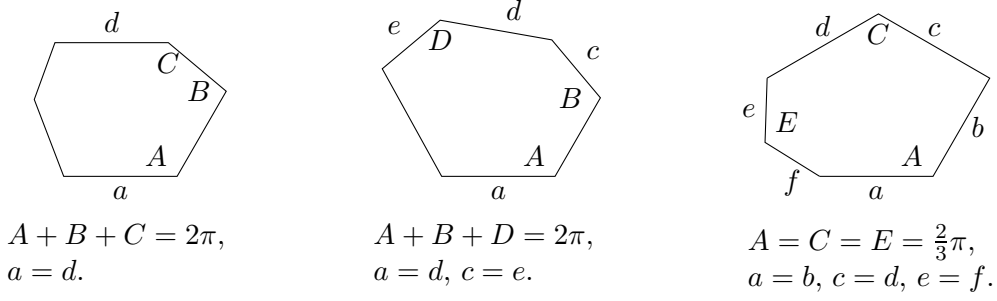


Figure 1. The 3 classes of hexagonal tilers.

Denote by $\mathcal{H}_{\mathcal{P}}$ the set of proper hexagonal cross-sections of \mathcal{P} .

Theorem 2.4 *Any proper hexagonal cross-section of a universal tiler, if exists, has a pair of opposite edges of the same length.*

Proof. Let \mathcal{P} be a universal tiler with $\mathcal{H}_{\mathcal{P}} \neq \emptyset$. For any $C \in \mathcal{H}_{\mathcal{P}}$, denote by $a(C)$ the number of angles of size $2\pi/3$ in C . Let

$$S = \{H \in \mathcal{H}_{\mathcal{P}} : \text{any pair of opposite edges of } H \text{ has distinct lengths}\}.$$

Suppose to the contrary that $S \neq \emptyset$. Let $H = V_1 V_2 \cdots V_6 \in S$ such that

$$a(H) = \min\{a(C) : C \in S\}.$$

By Theorem 2.3, we have $a(H) \geq 3$.

Without loss of generality, we can suppose that

$$\angle V_1 V_2 V_3 = \frac{2\pi}{3}. \quad (2.2)$$

By Lemma 2.2, there exists $\delta > 0$ such that for any $0 < \varepsilon < \delta$,

$$C(H; V_1 V_3; \varepsilon) = U_1^\varepsilon U_2^\varepsilon U_3^\varepsilon U_4^\varepsilon U_5^\varepsilon U_6^\varepsilon \in \mathcal{H}_{\mathcal{P}},$$

where $U_1^\varepsilon = V_1$, $U_3^\varepsilon = V_3$, $U_i^\varepsilon \in e_i$ and

$$\angle V_1 U_2^\varepsilon V_3 \neq \frac{2\pi}{3}. \quad (2.3)$$

On the other hand, by continuity, there exists $0 < \eta < \delta$ such that for any $i \bmod 6$,

$$\left| U_i^\eta U_{i+1}^\eta - U_{i+3}^\eta U_{i+4}^\eta \right| \geq \frac{1}{2} \left| V_i V_{i+1} - V_{i+3} V_{i+4} \right|, \quad (2.4)$$

$$\left| \angle U_i^\eta U_{i+1}^\eta U_{i+2}^\eta - \frac{2\pi}{3} \right| \geq \frac{1}{2} \left| \angle V_i V_{i+1} V_{i+2} - \frac{2\pi}{3} \right|. \quad (2.5)$$

Write $H^\eta = C(H; V_1 V_3; \eta)$. Then $H^\eta \in S$ by (2.4). In view of (2.2), (2.3) and (2.5), we deduce that $a(H^\eta) \leq a(H) - 1$, contradicting to the choice of H . This completes the proof. ■

As will be seen, we shall obtain that any universal tiler has no hexagonal cross-sections. But we need Theorem 2.4 to derive this result.

3 The valence-sets of universal tilers

In this section, we show that any facet of a universal tiler is either a triangle or a quadrangle. Let $F = V_1 V_2 \cdots V_n$ be a facet of \mathcal{P} . Let d_i be the valence of V_i . We say that the multiset $\{d_1, d_2, \dots, d_n\}$ is the valence-set of F . For example, the valence-set of any facet of a tetrahedron is $\{3, 3, 3\}$.

Lemma 3.1 *Let \mathcal{P} be a universal tiler. Let $\{d_1, d_2, \dots, d_n\}$ be a valence-set of a facet of \mathcal{P} . Then for any $1 \leq h \leq n$, there is a cross-section of \mathcal{P} with $\sum_{i=1}^n d_i - d_h - 2n + 4$ edges. Consequently, we have*

$$\sum_{i=1}^n d_i - d_h \leq 2n + 2. \quad (3.1)$$

Proof. Let $F = V_1 V_2 \cdots V_n$ be a facet of \mathcal{P} , where V_i has valence d_i . It suffices to show for the case $h = 1$. We shall prove by construction.

For convenience, we set up an xyz -coordinate system as follows. First, choose a point U_1 from the interior of the edge $V_n V_1$. Set U_1 to be the origin. Next, choose U_2 from the interior of $V_1 V_2$, and build the x -axis by putting U_2 on the positive x -axis. Then, build the y -axis such that F lies on the xy -plane and the y -coordinate of V_1 is negative. Consequently, all the other vertices V_2, \dots, V_n have positive y -coordinates. Since F is a facet, the convex polyhedron \mathcal{P} must lie entirely in one of the two half-spaces divided by the xy -plane. Build the z -axis such that all points in \mathcal{P} have nonnegative z -coordinates. Now we have an xyz -coordinate system.

Let $S = \{F' \mid F' \text{ is a facet of } \mathcal{P}, F' \cap F \neq \emptyset, F' \neq F\}$ with $|S| = s$. It is easy to see that

$$s = \sum_{i=1}^n d_i - 2n. \quad (3.2)$$

By continuity, there exists $\delta > 0$ such that for any $0 < \varepsilon < \delta$, the cross-section $C(z = \varepsilon)$ has exactly s vertices. Here, as usual, the equation $z = \varepsilon$ represents the plane parallel to the xy -plane with distance ε . Write

$$C(z = \varepsilon) = C_1^\varepsilon C_2^\varepsilon \cdots C_s^\varepsilon.$$

Then for any vertex C_j^ε , there is a unique vertex V_i such that V_i and C_j^ε lie in the same edge of \mathcal{P} . Denote this V_i by R_j^ε . Clearly R_j^ε is independent of ε . So we can omit the superscript ε and simply write R_j . Without loss of generality, we can suppose that

$$R_1 = R_2 = \cdots = R_t = V_1, \quad R_{t+1} = V_2, \quad R_s = V_n,$$

where

$$t = d_1 - 2. \quad (3.3)$$

Let $t + 1 \leq k \leq s$, and let y_k^ε be the y -coordinate of C_k^ε . Since V_2, \dots, V_n have positive y -coordinates, there exists $0 < z_0 < \delta$ such that $y_k^\varepsilon > 0$ for any $0 < \varepsilon \leq z_0$. For simplifying notation, we rewrite

$$C(z = z_0) = C_1 C_2 \cdots C_s.$$

Let y_k be the y -coordinate of C_k . Set

$$\varepsilon_0 = \frac{1}{2} \min \left\{ z_0, \frac{z_0}{y_{t+1}}, \frac{z_0}{y_{t+2}}, \dots, \frac{z_0}{y_s} \right\}. \quad (3.4)$$

We shall show that the cross-section $C(\pi_0)$ has $\sum_{i=2}^n d_i - 2n + 4$ edges. Consider the function f defined by

$$f(V) = \varepsilon_0 y - z,$$

where $V = (x, y, z)$ is a point. Denote by π_0 the plane determined by the equation $f(V) = 0$. On one hand, we have $f(R_k) > 0$ since the vertex R_k has positive y -coordinate and zero z -coordinate. On the other hand, by (3.4) we have

$$f(C_k) = \varepsilon_0 y_k - z_0 \leq \frac{1}{2} \cdot \frac{z_0}{y_k} \cdot y_k - z_0 < 0.$$

Therefore, the points R_k and C_k lie on distinct sides of π_0 . Consequently, the plane π_0 intersects the line segment $C_k R_k$. Let I_k be the intersecting point. Recall that U_1 is the origin and U_2 lies on the positive x -axis. So these two points belong to the plane π_0 . Hence

$$C(\pi_0) = U_1 U_2 I_{t+1} I_{t+2} \cdots I_s.$$

By (3.2) and (3.3), the number of edges of $C(\pi_0)$ is

$$s - t + 2 = \sum_{i=1}^n d_i - 2n - (d_1 - 2) + 2 = \sum_{i=2}^n d_i - 2n + 4.$$

Since any cross-section of a universal tiler has at most 6 edges, the inequality (3.1) follows immediately. This completes the proof. \blacksquare

Lemma 3.2 *The valence-set of any facet of a universal tiler is not $\{3, 3, 3, 3, 3\}$.*

Proof. Let \mathcal{P} be a universal tiler. Suppose to the contrary that \mathcal{P} has a pentagonal facet F whose every vertex has valence 3. For convenience, write $F = U_1' U_2 U_3 U_4 U_5$. Pick a point U_1 from the interior of the line segment $U_1' U_2$ such that

$$U_1 U_2 \neq U_4 U_5. \quad (3.5)$$

Pick a point U_6 from the interior of the line segment $U_5 U_1'$ such that

$$U_2 U_3 \neq U_5 U_6 \quad \text{and} \quad U_3 U_4 \neq U_6 U_1. \quad (3.6)$$

The existences of U_1 and U_6 are clear. Since the valence of each vertex of F is 3, there exists δ such that for any $0 < \varepsilon < \delta$,

$$C(F; U_6 U_1; \varepsilon) = U_1^\varepsilon U_2^\varepsilon U_3^\varepsilon U_4^\varepsilon U_5^\varepsilon U_6^\varepsilon \in \mathcal{H}_{\mathcal{P}},$$

where $U_1^\varepsilon = U_1$, $U_6^\varepsilon = U_6$, and U_i^ε and U_i lie on the same edge of \mathcal{P} for each $2 \leq i \leq 5$. On the other hand, by continuity, there exists $0 < \eta < \delta$ such that for any $i \bmod 6$,

$$\left| U_i^\eta U_{i+1}^\eta - U_{i+3}^\eta U_{i+4}^\eta \right| \geq \frac{1}{2} \left| U_i U_{i+1} - U_{i+3} U_{i+4} \right|. \quad (3.7)$$

In light of (3.5), (3.6) and (3.7), we see that the cross-section $C(F; U_1 U_6; \eta)$ has no pair of opposite edges of the same length, contradicting to Theorem 2.4. This completes the proof. \blacksquare

Using similar combinatorial arguments as in the above proof, we can determine the shape of a facet of a universal tiler.

Theorem 3.3 *Let \mathcal{P} be a universal tiler. Then every facet of \mathcal{P} is either a triangle or a quadrangle. Moreover, the valence-set of any triangular facet (if exists) of \mathcal{P} is either $\{4, 3, 3\}$ or $\{3, 3, 3\}$, while the valence-set of any quadrilateral facet (if exists) of \mathcal{P} is $\{3, 3, 3, 3\}$.*

Proof. Let $F_n = V_1V_2 \cdots V_n$ be a facet of \mathcal{P} . Let $S_n = \{d_1, d_2, \dots, d_n\}$ be the valence-set of F_n . By Lemma 3.1, we see that for any $1 \leq h \leq n$,

$$2n + 2 \geq \sum_{i=1}^n d_i - d_h \geq 3(n-1).$$

Namely $n \leq 5$. If $n = 5$, then (3.1) reads

$$\sum_{i=1}^5 d_i - d_h \leq 12.$$

Since each $d_i \geq 3$, we deduce that all $d_i = 3$, contradicting to Lemma 3.2. Hence $n \leq 4$.

If $n = 4$, then the valence-set S_4 is either $\{3, 3, 3, 3\}$ or $\{4, 3, 3, 3\}$ by (3.1). Assume that $S_4 = \{4, 3, 3, 3\}$, where V_1 has valence 4. Pick a point A from the interior of the line segment V_1V_2 such that $V_1A \neq V_3V_4$, and a point B from the interior of V_2V_3 such that $AB \neq V_4V_1$. Similar to the proof of Lemma 3.2, we can deduce that there exists η such that the cross-section $C(F_4; AB; \eta)$ belongs to $\mathcal{H}_{\mathcal{P}}$, and it has no pair of opposite edges of the same length, contradicting to Theorem 2.4. Hence $S_4 = \{3, 3, 3, 3\}$.

Consider the case $n = 3$. By Lemma 3.1, the valence-set S_3 has five possibilities:

$$\{3, 3, 3\}, \quad \{4, 3, 3\}, \quad \{4, 4, 3\}, \quad \{4, 4, 4\}, \quad \{5, 3, 3\}.$$

If $S_3 = \{4, 4, 3\}$ or $S_3 = \{4, 4, 4\}$, we can suppose that both V_1 and V_2 have valence 4. Pick a point A from the interior of V_1V_2 , and B from V_2V_3 such that $AB \neq V_3V_1$. Again, there exists η such that $C(F_3; AB; \eta)$ has no pair of opposite edges of the same length, contradicting to Theorem 2.4. If $S_3 = \{5, 3, 3\}$, we can suppose that V_1 has valence 5. Pick A from the interior of V_1V_2 such that $V_1A \neq V_3V_1$, and B from V_2V_3 . By similar arguments, we get a contradiction to Theorem 2.4. Hence the valence-set S_3 is either $\{3, 3, 3\}$ or $\{4, 3, 3\}$. This completes the proof. \blacksquare

4 The shapes of universal tilers

In this section, we show that every universal tiler at at most 5 facets.

Let \mathcal{P} be a universal tiler. Let f (resp. v , e) be the total number of facets (resp. vertices, edges) of \mathcal{P} . Euler's formula reads

$$f + v = e + 2. \tag{4.1}$$

It is well-known that there are two distinct topological types of pentahedra. One is the quadrilateral-based pyramids, which has the parameters

$$(v, e, f) = (5, 8, 5);$$

the other is pentahedra composed of two triangular bases and three quadrilateral sides, which has

$$(v, e, f) = (6, 9, 5).$$

Let f_i be the number of facets of i edges in \mathcal{P} . Let v_i be the number of vertices of valence i in \mathcal{P} . By Theorem 3.3, we have

$$f = f_3 + f_4 \quad \text{and} \quad v = v_3 + v_4. \quad (4.2)$$

Here is the main result of this paper.

Theorem 4.1 *A convex polyhedron is a universal tiler only if it is a tetrahedron or a pentahedron.*

Proof. Let \mathcal{P} be a universal tiler. By Theorem 3.3, every facet of \mathcal{P} has at most 4 edges and every vertex of \mathcal{P} has valence at most 4.

First, we deduce some relations by double-counting. Counting the pairs (e', f') where f' is a facet of \mathcal{P} and e' is an edge of f' , we find that

$$3f_3 + 4f_4 = 2e. \quad (4.3)$$

Counting the pairs (v', e') where e' is an edge of \mathcal{P} and v' is a vertex of e' , we obtain

$$3v_3 + 4v_4 = 2e. \quad (4.4)$$

Combining the relations from (4.1) to (4.4), we deduce that

$$(3f_3 + 4f_4) + (3v_3 + 4v_4) = 4e = 4(v + f - 2) = 4(v_3 + v_4 + f_3 + f_4 - 2),$$

namely

$$f_3 + v_3 = 8. \quad (4.5)$$

On the other hand, taking the difference of (4.3) and (4.4) yields

$$4(f_4 - v_4) = 3(v_3 - f_3). \quad (4.6)$$

Now we count the pairs (v', T) , where T is a triangular facet of \mathcal{P} and v' is a vertex of T having valence 4. By Theorem 3.3, every triangular facet has at most one vertex of valence 4, and every facet containing a vertex of valence 4 must be a triangle. Therefore

$$4v_4 \leq f_3. \quad (4.7)$$

By (4.5), (4.6) and (4.7), we deduce that $f_3 \leq 4$. Note that f_3 is an even number by (4.3). So $f_3 \in \{0, 2, 4\}$.

If $f_3 = 4$, then $v_3 = 4$ by (4.5), and $f_4 = v_4 \leq 1$ by (4.6) and (4.7). In this case, \mathcal{P} is a tetrahedron if $f_4 = 0$, and \mathcal{P} is a quadrilateral-based pyramid if $f_4 = 1$.

If $f_3 = 2$, then $v_3 = 6$ by (4.5), $v_4 = 0$ by (4.7), and consequently $f_4 = 3$ by (4.6). In this case, \mathcal{P} is a pentahedron composed of two triangular bases and three quadrilateral sides.

If $f_3 = 0$, then $v_3 = 8$, $v_4 = 0$, and $f_4 = 6$. Thus \mathcal{P} is a cube. We shall show that it is impossible. Denote

$$\mathcal{P} = ABCD-EFGH.$$

For convenience, we set up an xyz -coordinate system such that the plane $z=0$ coincides with the plane ACH , and the vertex D has negative z -coordinate. Let z_B (resp. z_E , z_F , z_G) be the z -coordinate of B (resp. E , F , G). Since \mathcal{P} is convex, all these z -coordinates are positive. Write

$$\delta = \frac{1}{2} \min\{z_B, z_E, z_F, z_G\}.$$

Then the line segment AB intersects the plane $z = \varepsilon$. Let A_1^ε be the intersecting point. Similarly, let A_2^ε (resp. C_1^ε , C_2^ε , H_1^ε , H_2^ε) be the intersection of the plane $z = \varepsilon$ and the line segment AE (resp. BC , CG , GH , HE). So

$$C(z=\varepsilon) = A_2^\varepsilon A_1^\varepsilon C_1^\varepsilon C_2^\varepsilon H_1^\varepsilon H_2^\varepsilon \in \mathcal{H}_{\mathcal{P}}.$$

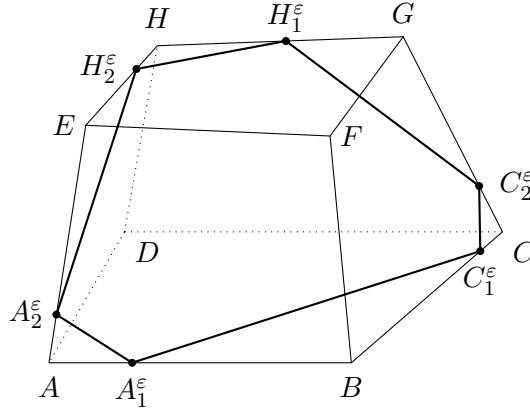


Figure 2. The hexagonal cross-section $C(z=\varepsilon) = A_2^\varepsilon A_1^\varepsilon C_1^\varepsilon C_2^\varepsilon H_1^\varepsilon H_2^\varepsilon$.

By continuity, we have

$$A_1^\varepsilon A_2^\varepsilon \rightarrow 0, \quad C_1^\varepsilon C_2^\varepsilon \rightarrow 0, \quad H_1^\varepsilon H_2^\varepsilon \rightarrow 0,$$

as $\varepsilon \rightarrow 0$. So there is $0 < \eta < \delta$ such that the cross-section $C(z=\eta)$ has no pair of opposite edges of the same length, contradicting to Theorem 2.4. This completes the proof. \blacksquare

Recall that any tetrahedron T is a universal tiler. We present that pentahedron universal tilers also exist.

Theorem 4.2 *Any pentahedron having a pair of parallel facets is a universal tiler.*

Proof. Suppose that \mathcal{P} is a pentahedron with a pair of parallel facets. Note that any cross-section of a pentahedron has at most 5 edges. It suffices to show that any pentagonal cross-section of \mathcal{P} tiles the plane. Let C be a pentagonal cross-section of \mathcal{P} . Then C has a pair of parallel edges. As pointed out by Reinhardt in [8], any pentagon with a pair of parallel edges is a tiler. This completes the proof. ■

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